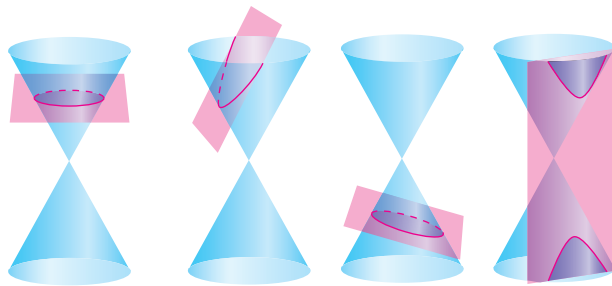# B Conic Sections

B.1 Conic Sections

- ■ Recognize the four basic conics: circles, parabolas, ellipses, and hyperbolas.
- ■ Recognize, graph, and write equations of parabolas (vertex at origin).
- ■ Recognize, graph, and write equations of ellipses (center at origin).
- ■ Recognize, graph, and write equations of hyperbolas (center at origin).

Introduction to Conic Sections

Conic sections were discovered during the classical Greek period, which lasted from 600 to 300 B.C. By the beginning of the Alexandrian period, enough was known of conics for Apollonius (262–190 B.C.) to produce an eight-volume work on the subject.

This early Greek study was largely concerned with the geometric properties of conics. It was not until the early seventeenth century that the broad applicability of conics became apparent.

A **conic section** (or simply **conic**) can be described as the intersection of a plane and a double-napped cone. Notice from Figure B.1 that in the formation of the four basic conics, the intersecting plane does not pass through the vertex of the cone. When the plane does pass through the vertex, the resulting figure is a **degenerate conic,** as shown in Figure B.2.

Conic Sections
FIGURE B.1
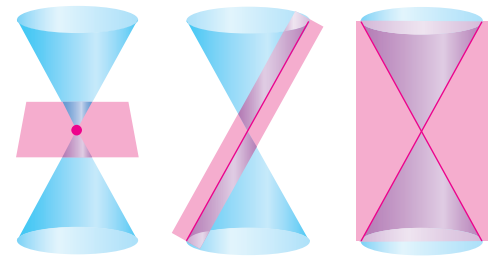
Degenerate Conics
FIGURE B.2

There are several ways to approach the study of conics. You could begin by defining conics in terms of the intersections of planes and cones, as the Greeks did, or you could define them algebraically, in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

However, you will study a third approach in which each of the conics is defined as a **locus,** or collection, of points satisfying a certain geometric property. For example, in Section 2.1 you saw how the definition of a circle as *the collection of all points (x, y) that are equidistant from a fixed point (h, k)* led easily to the standard equation of a circle,

$$(x - h)^2 + (y - k)^2 = r^2.$$

You will restrict your study of conics in Appendix B.1 to parabolas with vertices at the origin, and ellipses and hyperbolas with centers at the origin. In Appendix B.2, you will look at the general cases.
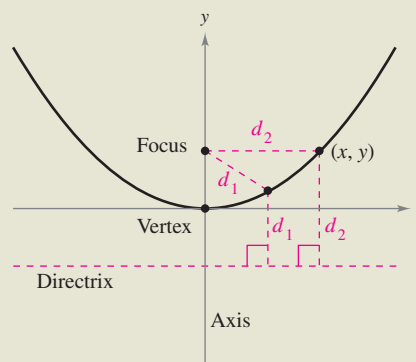
Parabolas

In Section 3.1, you determined that the graph of the quadratic function given by

$$f(x) = ax^2 + bx + c$$

is a parabola that opens upward or downward. The definition of a parabola given below is more general in the sense that it is independent of the orientation of the parabola.

Definition of a Parabola

A **parabola** is the set of all points (x, y) in a plane that are equidistant from a fixed line called the **directrix** and a fixed point called the **focus** (not on the line). The midpoint between the focus and the directrix is called the **vertex,** and the line passing through the focus and the vertex is called the **axis** of the parabola.

Using this definition, you can derive the following standard form of the equation of a parabola.

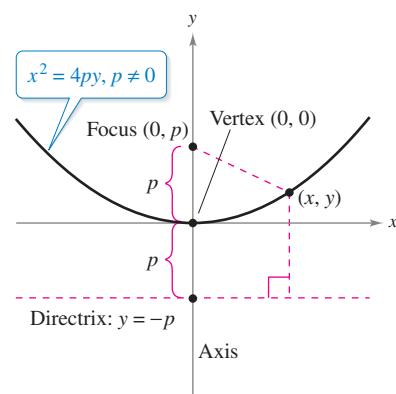### Standard Equation of a Parabola (Vertex at Origin)

The **standard form of the equation of a parabola** with vertex at $(0, 0)$ and directrix $y = -p$ is given by

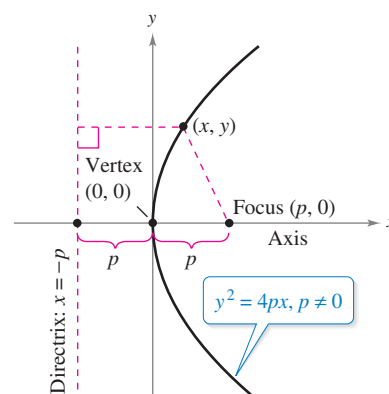$$x^2 = 4py, \quad p \neq 0. \qquad \text{Vertical axis}$$

For directrix $x = -p$, the equation is given by

$$y^2 = 4px, \quad p \neq 0. \qquad \text{Horizontal axis}$$

The focus is on the axis p units (directed distance) from the vertex. See Figure B.3.

Parabola with Vertical Axis

Parabola with Horizontal Axis

FIGURE B.3

Example 1   Finding the Focus of a Parabola

Find the focus of the parabola whose equation is $y = -2x^2$.

SOLUTION   Because the squared term in the equation involves x, you know that the axis is vertical, and the equation is of the form

$$x^2 = 4py. \qquad \text{Standard form, vertical axis}$$

You can write the original equation in this form, as shown.

$$-2x^2 = y \qquad \text{Write original equation.}$$

$$x^2 = -\frac{1}{2}y \qquad \text{Divide each side by } -2.$$

$$x^2 = 4\left(-\frac{1}{8}\right)y. \qquad \text{Write in standard form.}$$

So, $p = -\frac{1}{8}$. Because p is negative, the parabola opens downward and the focus of the parabola is $(0, p) = \left(0, -\frac{1}{8}\right)$, as shown in Figure B.4.

Focus $\left(0, -\frac{1}{8}\right)$
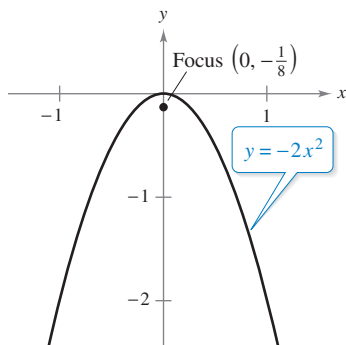
$y = -2x^2$

FIGURE B.4

✓ Checkpoint 1

Find the focus of the parabola whose equation is $y^2 = 2x$.   ■

Example 2   Finding the Standard Equation of a Parabola

Write the standard form of the equation of the parabola with vertex at the origin and focus at $(2, 0)$.

SOLUTION   The axis of the parabola is horizontal, passing through $(0, 0)$ and $(2, 0)$, as shown in Figure B.5. So, the standard form is

$$y^2 = 4px. \qquad \text{Standard form, horizontal axis}$$

Because the focus is $p = 2$ units from the vertex, the equation is

$$y^2 = 4(2)x \qquad \text{Standard form}$$
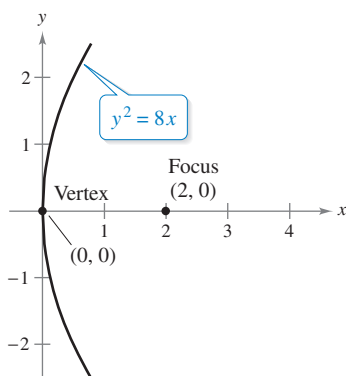
$$y^2 = 8x.$$

$y^2 = 8x$

Focus $(2, 0)$

Vertex

$(0, 0)$

FIGURE B.5

✓ Checkpoint 2

Write the standard form of the equation of the parabola with vertex at the origin and focus at $(0, 2)$.   ■

Parabolas occur in a wide variety of applications. For instance, a parabolic reflector can be formed by revolving a parabola about its axis. The resulting surface has the property that all incoming rays parallel to the axis are reflected through the focus of the parabola. This is the principle behind the construction of the parabolic mirrors used in reflecting telescopes. Conversely, the light rays emanating from the focus of the parabolic reflector used in a flashlight are all reflected parallel to one another, as shown in Figure B.6.
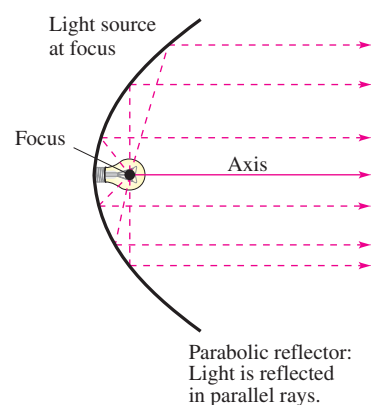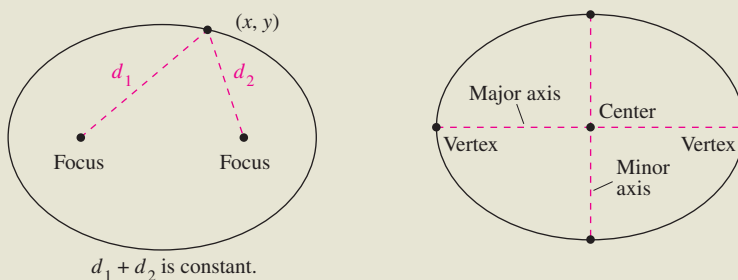

Light source at focus

Focus

Axis

Parabolic reflector: Light is reflected in parallel rays.

FIGURE B.6

Ellipses

Another basic type of conic is called an **ellipse.**

Definition of an Ellipse

An **ellipse** is the set of all points (x, y) in a plane the sum of whose distances from two distinct fixed points, called **foci,** is constant.

$d_1 + d_2$ is constant.

The line through the foci intersects the ellipse at two points, called the **vertices.** The chord joining the vertices is called the **major axis,** and its midpoint is called the **center** of the ellipse. The chord perpendicular to the major axis at the center is called the **minor axis** of the ellipse. The endpoints of the minor axis of an ellipse are commonly referred to as the **co-vertices.**

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure B.7. If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse.

The standard form of the equation of an ellipse takes one of two forms, depending on whether the major axis is horizontal or vertical.
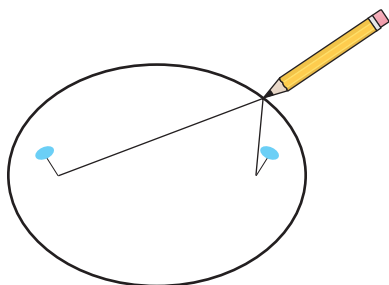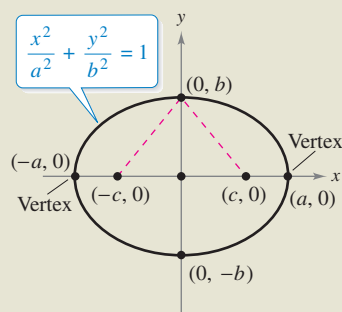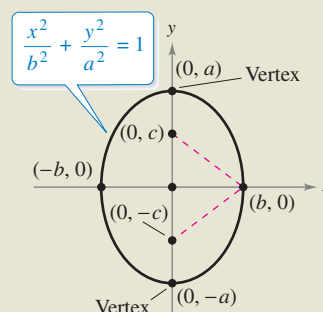

FIGURE B.7

Standard Equation of an Ellipse (Center at Origin)

The **standard form of the equation of an ellipse** with the center at the origin and major and minor axes of lengths $2a$ and $2b$, respectively (where $0 < b < a$), is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

Major axis is horizontal.
Minor axis is vertical.

Major axis is vertical.
Minor axis is horizontal.

The vertices and foci lie on the major axis, a and c units, respectively, from the center. Moreover, a, b, and c are related by the equation $c^2 = a^2 - b^2$.
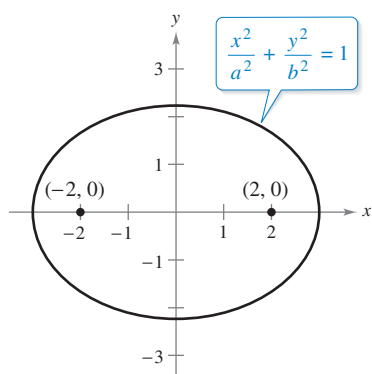
Example 3   Finding the Standard Equation of an Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(−2, 0)   (2, 0)

FIGURE B.8

Find the standard form of the equation of the ellipse that has a major axis of length 6 and foci at $(-2, 0)$ and $(2, 0)$, as shown in Figure B.8.

SOLUTION   Because the foci occur at $(-2, 0)$ and $(2, 0)$, the center of the ellipse is $(0, 0)$, and the major axis is horizontal. So, the ellipse has an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \text{Standard form, horizontal major axis}$$

Because the length of the major axis is 6, you have $2a = 6$, which implies that $a = 3$. Moreover, the distance from the center to either focus is $c = 2$. Finally, you have

$$b^2 = a^2 - c^2 = 3^2 - 2^2 = 9 - 4 = 5.$$

Substituting $a^2 = 3^2$ and $b^2 = \left(\sqrt{5}\right)^2$ yields the following equation in standard form.

$$\frac{x^2}{3^2} + \frac{y^2}{\left(\sqrt{5}\right)^2} = 1 \qquad \text{Standard form}$$

This equation simplifies to

$$\frac{x^2}{9} + \frac{y^2}{5} = 1.$$

✓ Checkpoint 3

Find the standard form of the equation of the ellipse that has a major axis of length 8 and foci at $(0, -3)$ and $(0, 3)$. ∎

Example 4   Sketching an Ellipse

Sketch the ellipse given by

$$4x^2 + y^2 = 36$$

and identify the vertices.

SOLUTION   Begin by writing the equation in standard form.

$$\frac{x^2}{3^2} + \frac{y^2}{6^2} = 1$$

(0, 6)   (−3, 0)   (3, 0)   (0, −6)

FIGURE B.9

$$4x^2 + y^2 = 36 \qquad \text{Write original equation.}$$

$$\frac{4x^2}{36} + \frac{y^2}{36} = \frac{36}{36} \qquad \text{Divide each side by 36.}$$

$$\frac{x^2}{9} + \frac{y^2}{36} = 1 \qquad \text{Simplify.}$$

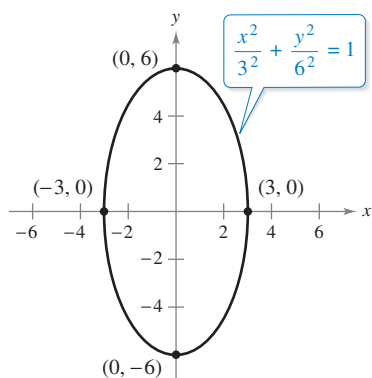$$\frac{x^2}{3^2} + \frac{y^2}{6^2} = 1 \qquad \text{Write in standard form.}$$

Because the denominator of the y^2-term is greater than the denominator of the x^2-term, you can conclude that the major axis is vertical. Also, because $a^2 = 6^2$, the endpoints of the major axis lie six units *up and down* from the center $(0, 0)$. So, the vertices of the ellipse are $(0, 6)$ and $(0, -6)$. Similarly, because the denominator of the x^2-term is $b^2 = 3^2$, the endpoints of the minor axis (or co-vertices) lie three units to the *right and left* of the center at $(3, 0)$ and $(-3, 0)$. The ellipse is shown in Figure B.9.

✓ Checkpoint 4

Sketch the ellipse given by $x^2 + 4y^2 = 64$, and identify the vertices. ∎

Hyperbolas

The definition of a **hyperbola** is similar to that of an ellipse. The distinction is that, for an ellipse, the *sum* of the distances between the foci and a point on the ellipse is constant, whereas for a hyperbola, the *difference* of the distances between the foci and a point on the hyperbola is constant.

Definition of a Hyperbola

A **hyperbola** is the set of all points (x, y) in a plane the difference of whose distances from two distinct fixed points, called **foci,** is constant.


$d_2 - d_1$ is constant.
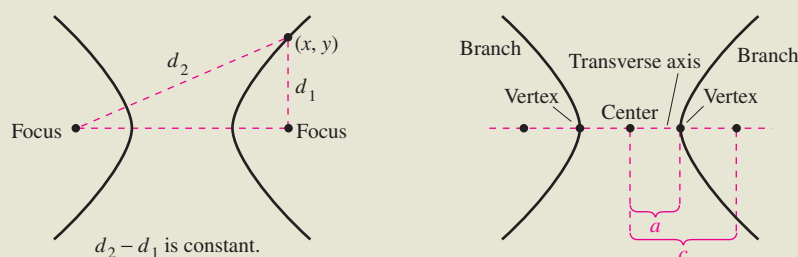
The graph of a hyperbola has two disconnected parts, called **branches.** The line through the two foci intersects the hyperbola at two points, called **vertices.** The line segment connecting the vertices is called the **transverse axis,** and the midpoint of the transverse axis is called the **center** of the hyperbola.

Standard Equation of a Hyperbola (Center at Origin)

The **standard form of the equation of a hyperbola** with the center at the origin (where $a \neq 0$ and $b \neq 0$) is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$
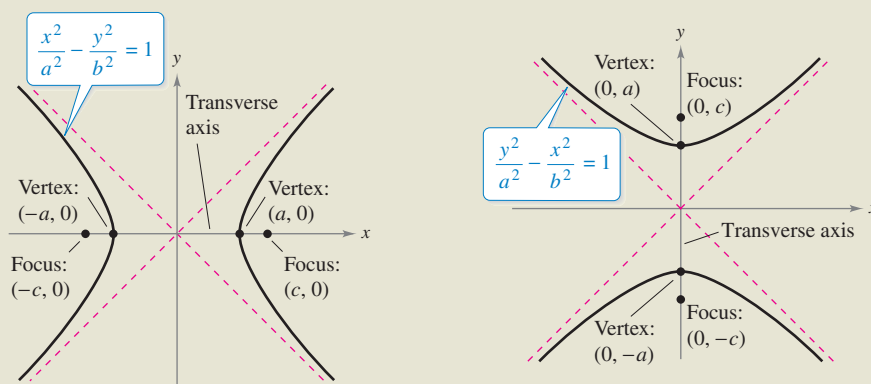
The vertices and foci are, respectively, a and c units from the center. Moreover, a, b, and c are related by the equation
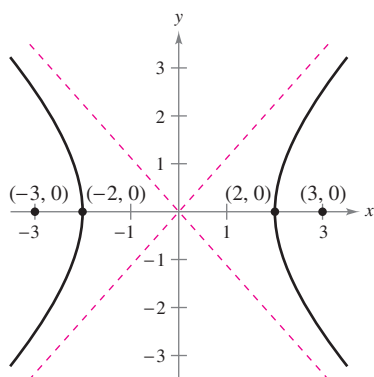
$$b^2 = c^2 - a^2.$$

Example 5 Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola with foci at $(-3, 0)$ and $(3, 0)$ and vertices at $(-2, 0)$ and $(2, 0)$, as shown in Figure B.10.

SOLUTION   From the graph, $c = 3$ because the foci are three units from the center. Also, $a = 2$ because the vertices are two units from the center. So,

$$b^2 = c^2 - a^2 = 3^2 - 2^2 = 9 - 4 = 5.$$

Because the transverse axis is horizontal, the standard form of the equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad \text{Standard form, horizontal transverse axis}$$

Finally, substituting $a^2 = 2^2$ and $b^2 = \left(\sqrt{5}\right)^2$, you have

$$\frac{x^2}{2^2} - \frac{y^2}{\left(\sqrt{5}\right)^2} = 1. \qquad \text{Write in standard form.}$$

FIGURE B.10

✓ **Checkpoint 5**

Find the standard form of the equation of the hyperbola with foci at $(0, -4)$ and $(0, 4)$ and vertices at $(0, -3)$ and $(0, 3)$. ■

An important aid in sketching the graph of a hyperbola is the determination of its *asymptotes*, as shown in Figure B.11. Each hyperbola has two asymptotes that intersect at the center of the hyperbola. Furthermore, the asymptotes pass through the corners of a rectangle of dimensions $2a$ by $2b$. The line segment of length $2b$, joining $(0, b)$ and $(0, -b)$ [or $(-b, 0)$ and $(b, 0)$], is referred to as the **conjugate axis** of the hyperbola.
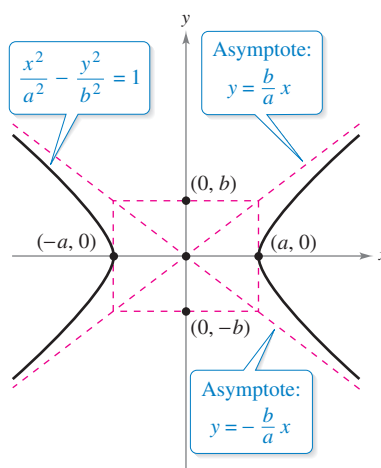
Transverse axis is horizontal.   Transverse axis is vertical.

FIGURE B.11
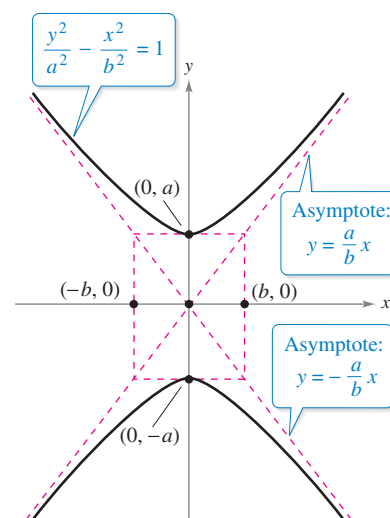
Asymptotes of a Hyperbola (Center at Origin)

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x \qquad \text{Transverse axis is horizontal.}$$

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x \qquad \text{Transverse axis is vertical.}$$

Example 6   Sketching a Hyperbola

Sketch the hyperbola whose equation is $4x^2 - y^2 = 16$.

SOLUTION

$$4x^2 - y^2 = 16$$ Write original equation.

$$\frac{4x^2}{16} - \frac{y^2}{16} = \frac{16}{16}$$ Divide each side by 16.

$$\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1$$ Write in standard form.

Because the x^2-term is positive, you can conclude that the transverse axis is horizontal and the vertices occur at $(-2, 0)$ and $(2, 0)$. Moreover, the endpoints of the conjugate axis occur at $(0, -4)$ and $(0, 4)$, and you can sketch the rectangle shown in Figure B.12. Finally, by drawing the asymptotes through the corners of this rectangle, you can complete the sketch shown in Figure B.13.
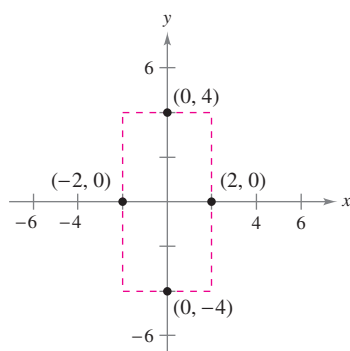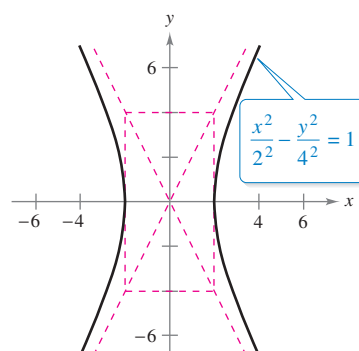

FIGURE B.12

FIGURE B.13

✓**Checkpoint 6**

Sketch the hyperbola whose equation is $9y^2 - x^2 = 9$.   ■

Example 7   Finding the Standard Equation of a Hyperbola

FIGURE B.14

Find the standard form of the equation of the hyperbola that has vertices at $(0, -3)$ and $(0, 3)$ and asymptotes $y = -2x$ and $y = 2x$, as shown in Figure B.14.

SOLUTION   Because the transverse axis is vertical, the asymptotes are of the form

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x.$$ Transverse axis is vertical.

Using the fact that $y = 2x$ and $y = -2x$, you can determine that $a/b = 2$. Because $a = 3$, you can determine that $b = \frac{3}{2}$. Finally, you can conclude that the hyperbola has the equation

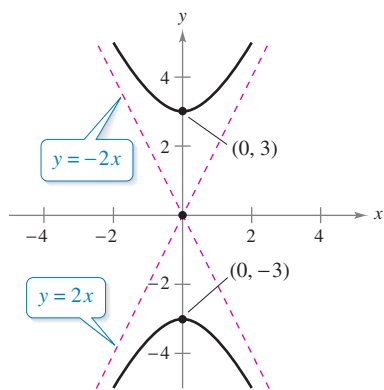$$\frac{y^2}{3^2} - \frac{x^2}{(3/2)^2} = 1.$$ Write in standard form.

✓**Checkpoint 7**

Find the standard form of the equation of the hyperbola that has vertices at $(-5, 0)$ and $(5, 0)$ and asymptotes $y = -x$ and $y = x$.   ■

In Exercises 1–4, rewrite the equation so that it has no fractions.

1. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

2. $\dfrac{x^2}{32} + \dfrac{4y^2}{32} = \dfrac{32}{32}$

3. $\dfrac{x^2}{1/4} - \dfrac{y^2}{4} = 1$

4. $\dfrac{3x^2}{1/9} + \dfrac{4y^2}{9} = 1$

In Exercises 5–8, solve for c. (Assume $c > 0$.)

5. $c^2 = 3^2 - 1^2$
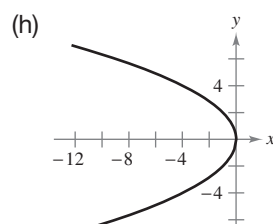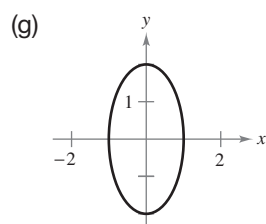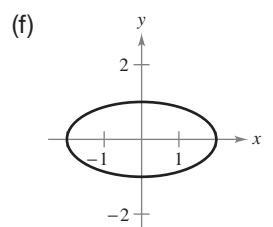
6. $c^2 = 2^2 + 3^2$

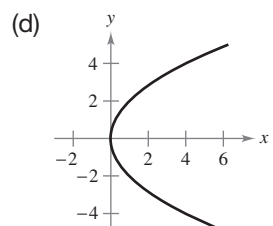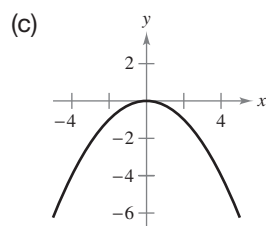**7.** $c^2 + 2^2 = 4^2$

8. $c^2 - 1^2 = 2^2$

In Exercises 9 and 10, find the distance between the point and the origin.

9. $(0, -4)$

10. $(-2, 0)$

Exercises B.1

Matching In Exercises 1–8, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]
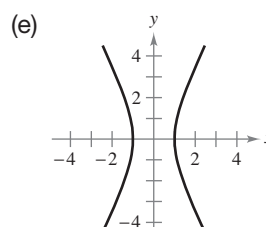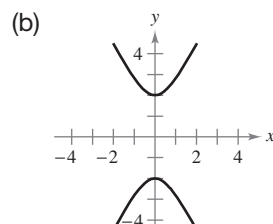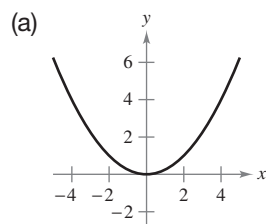
1. $x^2 = 4y$
2. $x^2 = -4y$
3. $y^2 = 4x$
4. $y^2 = -4x$
5. $\dfrac{x^2}{1} + \dfrac{y^2}{4} = 1$
6. $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$
7. $\dfrac{x^2}{1} - \dfrac{y^2}{4} = 1$
8. $\dfrac{y^2}{4} - \dfrac{x^2}{1} = 1$

(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)

Finding the Vertex and Focus of a Parabola In Exercises 9–16, find the vertex and focus of the parabola and sketch its graph. *See Example 1.*

9. $y = 4x^2$
10. $y = \frac{1}{2}x^2$
11. $y^2 = -6x$
12. $y^2 = 3x$
13. $x^2 + 8y = 0$
14. $x^2 + 12y = 0$
15. $x + y^2 = 0$
16. $y^2 - 8x = 0$

Finding the Standard Equation of a Parabola In Exercises 17–26, find the standard form of the equation of the parabola with the given characteristic(s) and vertex at the origin. *See Example 2.*

17. Focus: $\left(0, -\frac{3}{2}\right)$
18. Focus: $(0, -2)$

19. Focus: $(-2, 0)$

20. Focus: $\left(\frac{5}{2}, 0\right)$

21. Directrix: $y = -1$

22. Directrix: $y = 2$

23. Directrix: $x = 3$

24. Directrix: $x = -2$

25. Passes through the point $(4, 6)$; horizontal axis

26. Passes through the point $(-2, -2)$; vertical axis

Finding the Standard Equation of an Ellipse In Exercises 27–34, find the standard form of the equation of the ellipse with the given characteristics and center at the origin. *See Example 3.*

27. Vertices: $(0, \pm2)$; minor axis of length 2

28. Vertices: $(\pm2, 0)$; minor axis of length 3

29. Vertices: $(\pm5, 0)$; foci: $(\pm2, 0)$

30. Vertices: $(0, \pm10)$; foci: $(0, \pm4)$

31. Foci: $(\pm5, 0)$; major axis of length 12

32. Foci: $(\pm2, 0)$; major axis of length 8

33. Vertices: $(0, \pm5)$; passes through the point $(4, 2)$

34. Major axis vertical; passes through the points $(0, 4)$ and $(2, 0)$

Sketching an Ellipse In Exercises 35–42, find the center and vertices of the ellipse and sketch its graph. *See Example 4.*

35. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

36. $\dfrac{x^2}{144} + \dfrac{y^2}{169} = 1$

37. $\dfrac{x^2}{\frac{25}{9}} + \dfrac{y^2}{\frac{16}{9}} = 1$

38. $\dfrac{x^2}{4} + \dfrac{y^2}{\frac{1}{4}} = 1$

39. $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$

40. $\dfrac{x^2}{28} + \dfrac{y^2}{64} = 1$

41. $5x^2 + 3y^2 = 15$

42. $x^2 + 4y^2 = 4$

Finding the Standard Equation of a Hyperbola In Exercises 43–46, find the standard form of the equation of the hyperbola with the given characteristics and center at the origin. *See Example 5.*

43. Vertices: $(0, \pm2)$; foci: $(0, \pm4)$

44. Vertices: $(0, \pm5)$; foci: $(0, \pm8)$

45. Vertices: $(\pm6, 0)$; foci: $(\pm9, 0)$

46. Vertices: $(\pm3, 0)$; foci: $(\pm5, 0)$

Sketching a Hyperbola In Exercises 47–54, find the center and vertices of the hyperbola and sketch its graph. *See Example 6.*

47. $x^2 - y^2 = 1$

48. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

49. $\dfrac{y^2}{1} - \dfrac{x^2}{4} = 1$

50. $\dfrac{y^2}{9} - \dfrac{x^2}{1} = 1$

51. $\dfrac{y^2}{25} - \dfrac{x^2}{144} = 1$

52. $\dfrac{x^2}{36} - \dfrac{y^2}{4} = 1$

**53.** $2x^2 - 3y^2 = 6$

54. $3y^2 - 5x^2 = 15$

Finding the Standard Equation of a Hyperbola In Exercises 55–60, find the standard form of the equation of the hyperbola with the given characteristics and center at the origin. *See Example 7.*

55. Vertices: $(\pm1, 0)$; asymptotes: $y = \pm3x$

56. Vertices: $(0, \pm3)$; asymptotes: $y = \pm3x$

57. Foci: $(0, \pm4)$; asymptotes: $y = \pm\frac{1}{2}x$

58. Foci: $(\pm10, 0)$; asymptotes: $y = \pm\frac{3}{4}x$

59. Vertices: $(0, \pm3)$; passes through the point $(-2, 5)$

**60.** Vertices: $(\pm2, 0)$; passes through the point $\left(3, \sqrt{3}\right)$

61. Satellite Antenna The receiver in a parabolic television dish antenna is 3 feet from the vertex and is located at the focus (see figure). Write an equation for a cross section of the reflector. (Assume that the dish is directed upward and the vertex is at the origin.)
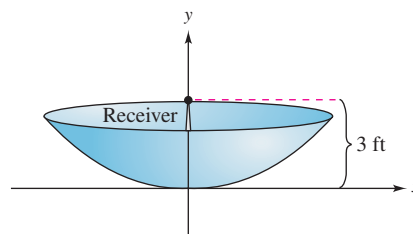
62. Suspension Bridge Each cable of the Golden Gate Bridge is suspended (in the shape of a parabola) between two towers that are 1280 meters apart. The top of each tower is 152 meters above the roadway (see figure). The cables touch the roadway at the midpoint between the towers. Write an equation for the parabolic shape of each cable.
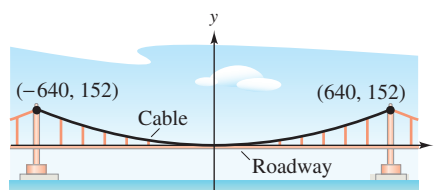
63. Architecture A fireplace arch is to be constructed in the shape of a semiellipse. The opening is to have a height of 2 feet at the center and a width of 5 feet along the base (see figure). The contractor draws the outline of the ellipse by the method shown in Figure B.7. Where should the tacks be placed and what should be the length of the piece of string?
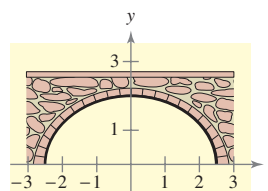
64. Mountain Tunnel A semielliptical arch over a tunnel for a road through a mountain has a major axis of 100 feet, and its height at the center is 30 feet (see figure). Determine the height of the arch 5 feet from the edge of the tunnel.
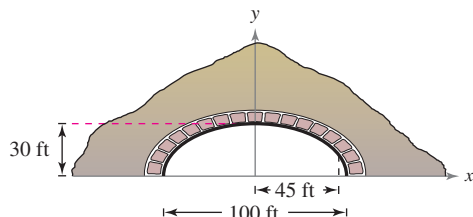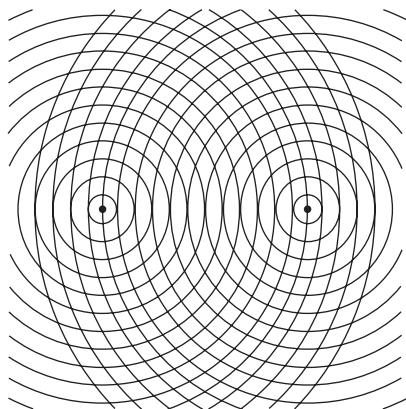

65. Sketching an Ellipse Sketch a graph of the ellipse that consists of all points (x, y) such that the sum of the distances between (x, y) and two fixed points is 15 units and the foci are located at the centers of the two sets of concentric circles, as shown in the figure.

66. Think About It A line segment through a focus of an ellipse with endpoints on the ellipse and perpendicular to its major axis is called a **latus rectum** of the ellipse. An ellipse has two latera recta. Knowing the length of the latera recta is helpful in sketching an ellipse because this information yields other points on the curve (see figure). Show that the length of each latus rectum is

$$\frac{2b^2}{a}.$$
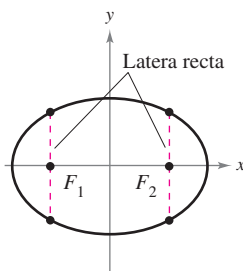
Using Latera Recta In Exercises 67–70, sketch the ellipse using the latera recta (see Exercise 66).

67. $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$ **68.** $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

69. $9x^2 + 4y^2 = 36$

70. $5x^2 + 3y^2 = 15$

**71. Think About It** Consider the ellipse

$$\frac{x^2}{328} + \frac{y^2}{327} = 1.$$

Is this ellipse better described as *elongated* or *nearly circular*? Explain your reasoning.

72. Navigation Long-range navigation for aircraft and ships is accomplished by synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci. Assume that two stations 300 miles apart are positioned on a rectangular coordinate system at points with coordinates $(-150, 0)$ and $(150, 0)$ and that a ship is traveling on a path with coordinates $(x, 75)$ (see figure). Find the x-coordinate of the position of the ship if the time difference between the pulses from the transmitting stations is 1000 microseconds (0.001 second).
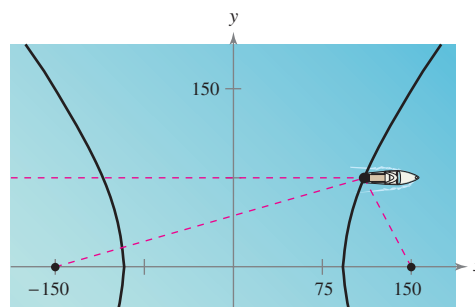
73. Hyperbolic Mirror A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at one focus will be reflected to the other focus (see figure). The focus of the hyperbolic mirror has coordinates $(12, 0)$. Find the vertex of the mirror if its mount at the top edge of the mirror has coordinates $(12, 12)$.
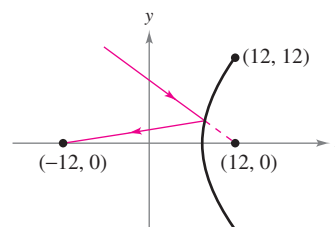
74. Writing Explain how the central rectangle of a hyperbola can be used to sketch its asymptotes.

B.2 Conic Sections and Translations

■ Recognize equations of conics that have been shifted vertically or horizontally in the plane.

■ Write and graph equations of conics that have been shifted vertically or horizontally in the plane.

Vertical and Horizontal Shifts of Conics

In Appendix B.1, you studied conic sections whose graphs were in **standard position.** In this section, you will study the equations of conic sections that have been shifted vertically or horizontally in the plane. The following summary lists the standard forms of the equations of the four basic conics.

$(x - h)^2 + (y - k)^2 = r^2$
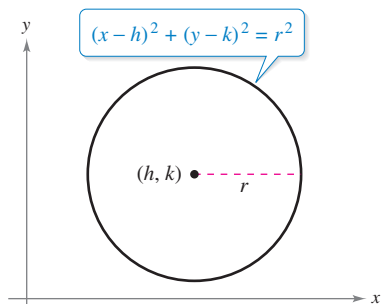
FIGURE B.15

Standard Forms of Equations of Conics

Circle: Center $= (h, k)$; Radius $= r$; See Figure B.15.

Parabola: Vertex $= (h, k)$; Directed distance from vertex to focus $= p$

$(x - h)^2 = 4p(y - k)$

$p > 0$

Vertex: (h, k)

Focus: $(h, k + p)$

$(y - k)^2 = 4p(x - h)$

$p > 0$

Vertex: (h, k)

Focus: $(h + p, k)$
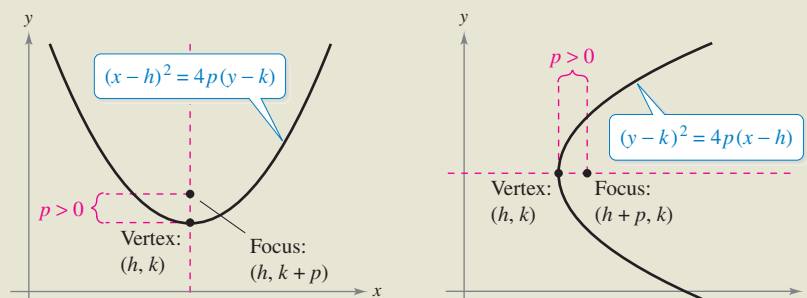
Ellipse: Center $= (h, k)$; Major axis length $= 2a$; Minor axis length $= 2b$

$\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$

(h, k) $2b$ $2a$

$\dfrac{(x - h)^2}{b^2} + \dfrac{(y - k)^2}{a^2} = 1$

(h, k) $2a$ $2b$
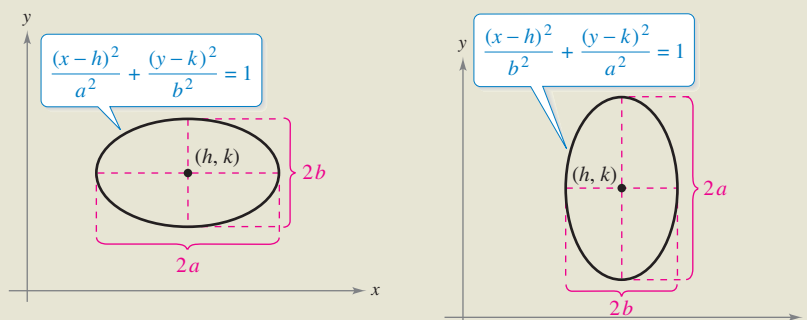
Hyperbola: Center $= (h, k)$; Transverse axis length $= 2a$; Conjugate axis length $= 2b$

$\dfrac{(x - h)^2}{a^2} - \dfrac{(y - k)^2}{b^2} = 1$

(h, k) $2b$ $2a$

$\dfrac{(y - k)^2}{a^2} - \dfrac{(x - h)^2}{b^2} = 1$

(h, k) $2a$ $2b$
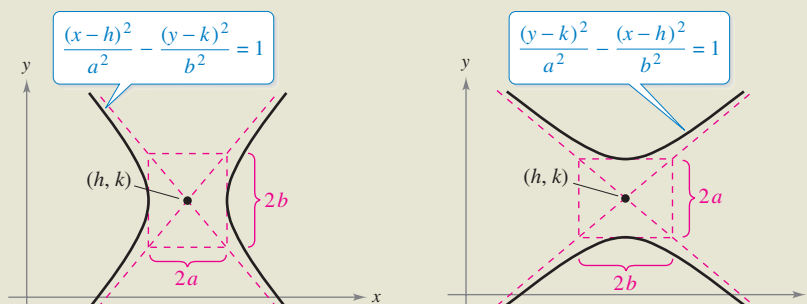
Example 1   Equations of Conic Sections

Describe the translation of the graph of each conic.

a. $(x - 1)^2 + (y + 2)^2 = 3^2$ **b.** $(x - 2)^2 = 4(-1)(y - 3)$

c. $\dfrac{(x - 3)^2}{1^2} - \dfrac{(y - 2)^2}{3^2} = 1$ **d.** $\dfrac{(x - 2)^2}{3^2} + \dfrac{(y - 1)^2}{2^2} = 1$

SOLUTION

a. The graph of $(x - 1)^2 + (y + 2)^2 = 3^2$ is a circle whose center is the point $(1, -2)$ and whose radius is 3, as shown in Figure B.16. The graph of the circle has been shifted one unit to the right and two units downward from standard position.

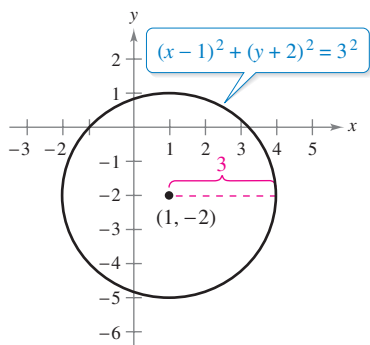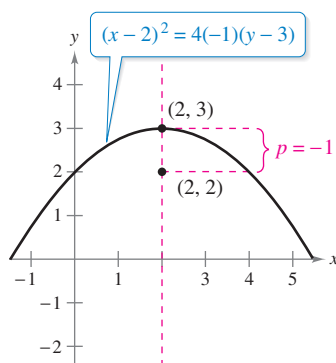**b.** The graph of $(x - 2)^2 = 4(-1)(y - 3)$ is a parabola whose vertex is the point $(2, 3)$. The axis of the parabola is vertical. Moreover, because $p = -1$, it follows that the focus lies *below* the vertex, as shown in Figure B.17. The graph of the parabola has been reflected in the x-axis, and shifted two units to the right and three units upward from standard position.

c. The graph of

$$\frac{(x - 3)^2}{1^2} - \frac{(y - 2)^2}{3^2} = 1$$

is a hyperbola whose center is the point $(3, 2)$. The transverse axis is horizontal with a length of $2(1) = 2$. The conjugate axis is vertical with a length of $2(3) = 6$, as shown in Figure B.18. The graph of the hyperbola has been shifted three units to the right and two units upward from standard position.

d. The graph of

$$\frac{(x - 2)^2}{3^2} + \frac{(y - 1)^2}{2^2} = 1$$

is an ellipse whose center is the point $(2, 1)$. The major axis of the ellipse is horizontal with a length of $2(3) = 6$. The minor axis of the ellipse is vertical with a length of $2(2) = 4$, as shown in Figure B.19. The graph of the ellipse has been shifted two units to the right and one unit upward from standard position.
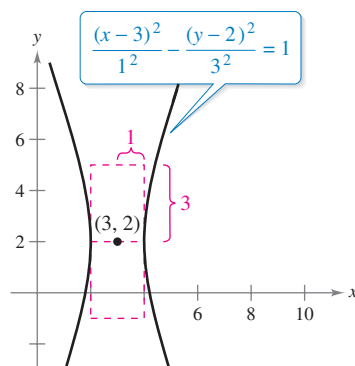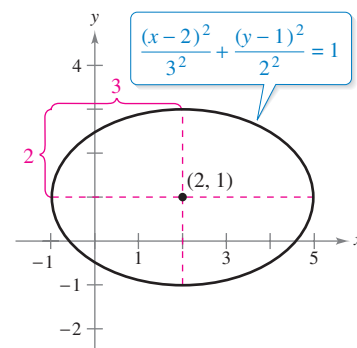

FIGURE B.16

FIGURE B.17

FIGURE B.18

FIGURE B.19

✓ Checkpoint 1

Describe the translation of the graph of

$$(x + 4)^2 + (y - 3)^2 = 5^2.$$

Writing Equations of Conics in Standard Form

Example 2   Finding the Standard Form of a Parabola

Find the vertex and focus of the parabola given by $x^2 - 2x + 4y - 3 = 0$.

SOLUTION   Complete the square to write the equation in standard form.

$$x^2 - 2x + 4y - 3 = 0 \qquad \text{Write original equation.}$$

$$x^2 - 2x = -4y + 3 \qquad \text{Group terms.}$$

$$x^2 - 2x + 1 = -4y + 3 + 1 \qquad \text{Complete the square.}$$

$$(x - 1)^2 = -4y + 4 \qquad \text{Write in completed square form.}$$

$$(x - 1)^2 = 4(-1)(y - 1) \qquad \text{Standard form, } (x - h)^2 = 4p(y - k)$$

FIGURE B.20

From this standard form, it follows that $h = 1$, $k = 1$, and $p = -1$. Because the axis is vertical and p is negative, the parabola opens downward. The vertex is $(h, k) = (1, 1)$ and the focus is $(h, k + p) = (1, 0)$. (See Figure B.20.)
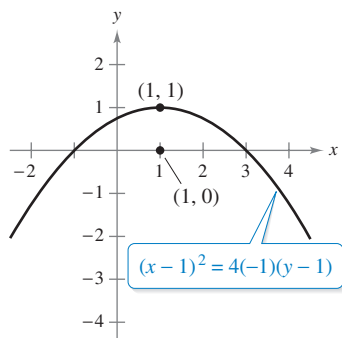
✓ Checkpoint 2

Find the vertex and focus of the parabola given by $x^2 + 2x - 4y + 5 = 0$.   ■

In Examples 1(b) and 2, p is the *directed distance* from the vertex to the focus. Because the axis of the parabola is vertical and $p = -1$, the focus is one unit *below* the vertex, and the parabola opens downward.

Example 3   Sketching an Ellipse

Sketch the ellipse given by $x^2 + 4y^2 + 6x - 8y + 9 = 0$.

SOLUTION   Complete the square to write the equation in standard form.

$$x^2 + 4y^2 + 6x - 8y + 9 = 0 \qquad \text{Write original equation.}$$

$$\left(x^2 + 6x + \boxed{}\right) + \left(4y^2 - 8y + \boxed{}\right) = -9 \qquad \text{Group terms.}$$

$$\left(x^2 + 6x + \boxed{}\right) + 4\left(y^2 - 2y + \boxed{}\right) = -9 \qquad \text{Factor 4 out of } y\text{-terms.}$$

$$(x^2 + 6x + 9) + 4(y^2 - 2y + 1) = -9 + 9 + 4(1) \qquad \text{Complete the squares.}$$

$$(x + 3)^2 + 4(y - 1)^2 = 4 \qquad \text{Write in completed square form.}$$

$$\frac{(x + 3)^2}{4} + \frac{(y - 1)^2}{1} = 1 \qquad \text{Divide each side by 4.}$$

$$\frac{(x + 3)^2}{2^2} + \frac{(y - 1)^2}{1^2} = 1 \qquad \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

From this standard form, it follows that the center is $(h, k) = (-3, 1)$. Because the denominator of the x-term is $a^2 = 2^2$, the endpoints of the major axis lie two units to the right and left of the center. Similarly, because the denominator of the y-term is $b^2 = 1^2$, the endpoints of the minor axis lie one unit up and down from the center. The ellipse is shown in Figure B.21.
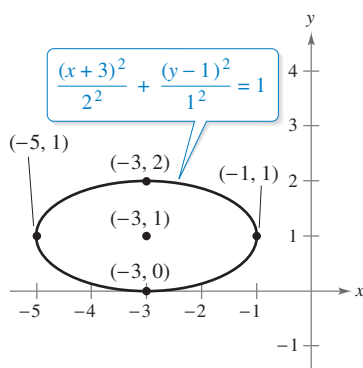
FIGURE B.21

✓ Checkpoint 3

Sketch the ellipse given by $x^2 + 9y^2 - 4x + 18y + 4 = 0$.   ■

Example 4   Sketching a Hyperbola

Sketch the hyperbola given by

$$y^2 - 4x^2 + 4y + 24x - 41 = 0.$$

SOLUTION   Complete the square to write the equation in standard form.

$$y^2 - 4x^2 + 4y + 24x - 41 = 0 \qquad \text{Write original equation.}$$

$$\left(y^2 + 4y + \boxed{}\right) - \left(4x^2 - 24x + \boxed{}\right) = 41 \qquad \text{Group terms.}$$

$$\left(y^2 + 4y + \boxed{}\right) - 4\left(x^2 - 6x + \boxed{}\right) = 41 \qquad \text{Factor 4 out of } x\text{-terms.}$$

$$(y^2 + 4y + 4) - 4(x^2 - 6x + 9) = 41 + 4 - 4(9) \qquad \text{Complete the squares.}$$

$$(y + 2)^2 - 4(x - 3)^2 = 9 \qquad \text{Write in completed square form.}$$

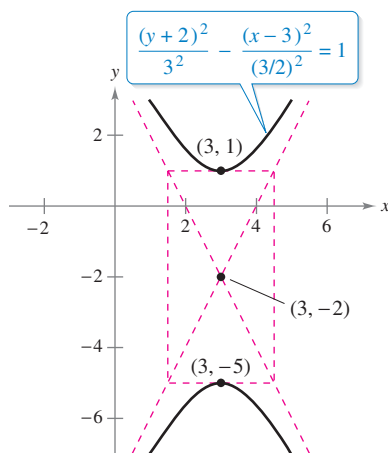$$\frac{(y + 2)^2}{9} - \frac{4(x - 3)^2}{9} = 1 \qquad \text{Divide each side by 9.}$$

$$\frac{(y + 2)^2}{9} - \frac{(x - 3)^2}{9/4} = 1 \qquad \text{Change 4 to } \frac{1}{1/4}.$$

$$\frac{(y + 2)^2}{3^2} - \frac{(x - 3)^2}{(3/2)^2} = 1 \qquad \frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

$$\frac{(y+2)^2}{3^2} - \frac{(x-3)^2}{(3/2)^2} = 1$$

(3, 1)

(3, −2)

(3, −5)

FIGURE B.22

From the standard form, it follows that the transverse axis is vertical and the center lies at $(h, k) = (3, -2)$. Because the denominator of the y-term is $a^2 = 3^2$, you know that the vertices lie three units above and below the center.

Vertices: $(3, 1)$ and $(3, -5)$

To sketch the hyperbola, draw a rectangle whose top and bottom pass through the vertices. Because the denominator of the x-term is $b^2 = \left(\frac{3}{2}\right)^2$, locate the sides of the rectangle $\frac{3}{2}$ units to the right and left of the center, as shown in Figure B.22. Finally, sketch the asymptotes by drawing lines through the opposite corners of the rectangle. Using these asymptotes, you can complete the graph of the hyperbola, as shown in Figure B.22.

✓ Checkpoint 4

Sketch the hyperbola given by

$$4x^2 - y^2 + 16x - 2y - 1 = 0.$$ ■

To find the foci in Example 4, first find c. Recall from Appendix B.1 that $b^2 = c^2 - a^2$. So,

$$c^2 = a^2 + b^2$$

$$c^2 = 9 + \frac{9}{4}$$

$$c^2 = \frac{45}{4}$$

$$c = \frac{3\sqrt{5}}{2}.$$

Because the transverse axis is vertical, the foci lie c units above and below the center.

$$\text{Foci: } \left(3, -2 + \frac{3\sqrt{5}}{2}\right) \quad \text{and} \quad \left(3, -2 - \frac{3\sqrt{5}}{2}\right)$$

FIGURE B.23

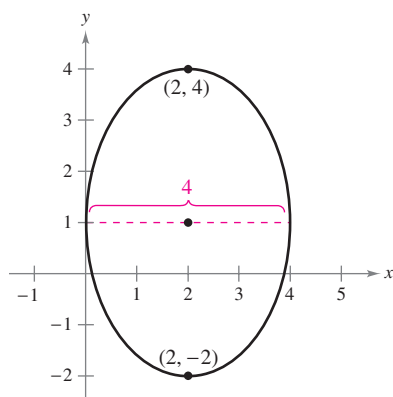### *Example 5* Writing the Equation of an Ellipse

Write the standard form of the equation of the ellipse whose vertices are $(2, -2)$ and $(2, 4)$. The length of the minor axis of the ellipse is 4, as shown in Figure B.23.

SOLUTION The center of the ellipse lies at the midpoint of its vertices. So, the center is

$$(h, k) = \left(\frac{2 + 2}{2}, \frac{4 + (-2)}{2} \right)$$

$$= (2, 1). \qquad \text{Center}$$

Because the vertices lie on a vertical line and are six units apart, it follows that the major axis is vertical and has a length of $2a = 6$. So, $a = 3$. Moreover, because the minor axis has a length of 4, it follows that $2b = 4$, which implies that $b = 2$. So, you can conclude that the standard form of the equation of the ellipse is

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1 \qquad \text{Major axis is vertical.}$$

$$\frac{(x - 2)^2}{2^2} + \frac{(y - 1)^2}{3^2} = 1. \qquad \text{Write in standard form.}$$

✓ Checkpoint 5

Write the standard form of the equation of the ellipse whose vertices are $(-1, 4)$ and $(4, 4)$. The length of the minor axis of the ellipse is 3. ■
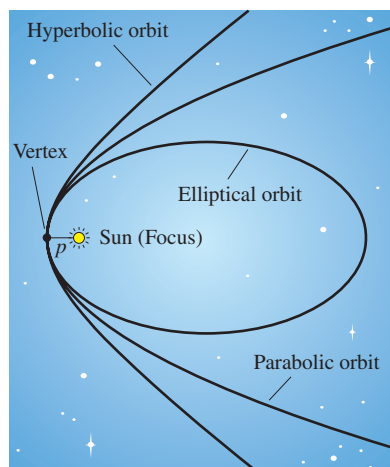

FIGURE B.24

An interesting application of conic sections involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. For example, Halley's comet has an elliptical orbit, and reappearance of this comet can be predicted every 76 years. The center of the sun is a focus of each of these orbits, and each orbit has a vertex at the point where the comet is closest to the sun, as shown in Figure B.24.

If p is the distance between the vertex and the focus (in meters), and v is the speed of the comet at the vertex (in meters per second), then the type of orbit is determined as follows.

1. Ellipse: $v < \sqrt{\dfrac{2GM}{p}}$

2. Parabola: $v = \sqrt{\dfrac{2GM}{p}}$

3. Hyperbola: $v > \sqrt{\dfrac{2GM}{p}}$

In these expressions, $M = 1.989 \times 10^{30}$ kilograms (the mass of the sun) and $G \approx 6.67 \times 10^{-11}$ cubic meter per kilogram-second squared (the universal gravitational constant).

In Exercises 1–10, identify the conic represented by the equation.

1. $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 1$

2. $\dfrac{x^2}{9} + \dfrac{y^2}{1} = 1$

3. $2x + y^2 = 0$

4. $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$

5. $4x^2 + 4y^2 = 25$

6. $\dfrac{y^2}{4} - \dfrac{x^2}{2} = 1$

7. $x^2 - 6y = 0$
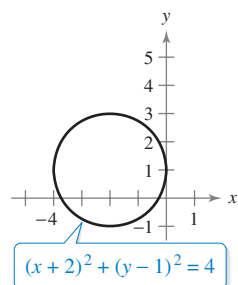
8. $3x - y^2 = 0$

9. $\dfrac{9x^2}{16} + \dfrac{y^2}{4} = 1$
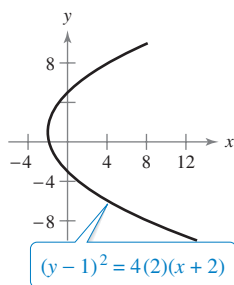
10. $3y^2 - 3x^2 = 48$

Exercises B.2

Equations of Conic Sections In Exercises 1–6, describe the translation of the graph of the conic from the standard position. *See Example 1.*
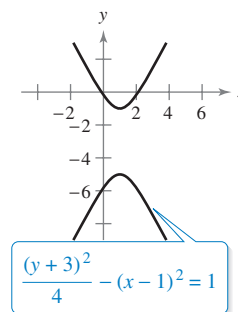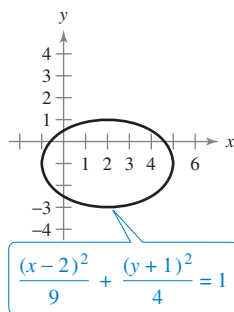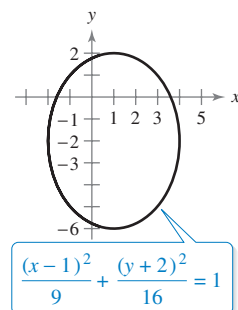
1.

$(x+2)^2 + (y-1)^2 = 4$

2.

$(y-1)^2 = 4(2)(x+2)$

3.

$\dfrac{(y+3)^2}{4} - (x-1)^2 = 1$

4.

$\dfrac{(x-2)^2}{9} + \dfrac{(y+1)^2}{4} = 1$

5.

$\dfrac{(x-1)^2}{9} + \dfrac{(y+2)^2}{16} = 1$
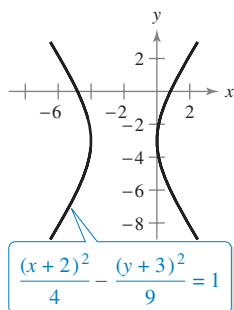
6.

$\dfrac{(x+2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$

Finding the Standard Form of a Parabola In Exercises 7–16, find the vertex, focus, and directrix of the parabola. Then sketch its graph. *See Example 2.*

7. $(x - 1)^2 + 8(y + 2) = 0$

8. $(x + 3) + (y - 2)^2 = 0$

9. $\left(y + \tfrac{1}{2}\right)^2 = 2(x - 5)$

10. $\left(x + \tfrac{1}{2}\right)^2 = 4(y - 3)$

11. $y = \tfrac{1}{4}(x^2 - 2x + 5)$

12. $y = -\tfrac{1}{6}(x^2 + 4x - 2)$

13. $4x - y^2 - 2y - 33 = 0$

14. $y^2 + x + y = 0$

15. $y^2 + 6y + 8x + 25 = 0$

16. $x^2 - 2x + 8y + 9 = 0$

Finding the Standard Form of a Parabola In Exercises 17–24, find the standard form of the equation of the parabola with the given characteristics.

17. Vertex: $(3, 2)$; focus: $(1, 2)$

18. Vertex: $(-1, 2)$; focus: $(-1, 0)$

19. Vertex: $(0, 4)$;
directrix: $y = 2$

20. Vertex: $(-2, 1)$;
directrix: $x = 1$

21. Focus: $(2, 2)$;
directrix: $x = -2$

22. Focus: $(0, 0)$;
directrix: $y = 4$

23. Vertex: $(0, 4)$;
passes through $(-2, 0)$ and $(2, 0)$

24. Vertex: $(2, 4)$;
passes through $(0, 0)$ and $(4, 0)$

Sketching an Ellipse   In Exercises 25–32, find the center, foci, and vertices of the ellipse. Then sketch its graph. *See Example 3.*

25. $\dfrac{(x-1)^2}{9} + \dfrac{(y-5)^2}{25} = 1$

26. $(x+2)^2 + \dfrac{(y+4)^2}{1/4} = 1$

27. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

28. $9x^2 + 4y^2 - 36x + 8y + 31 = 0$

29. $16x^2 + 25y^2 - 32x + 50y + 16 = 0$

30. $9x^2 + 25y^2 - 36x - 50y + 61 = 0$

31. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

32. $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

Sketching a Hyperbola   In Exercises 33–42, find the center, vertices, and foci of the hyperbola. Then sketch its graph, using asymptotes as sketching aids. *See Example 4.*

33. $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{1} = 1$

34. $\dfrac{(x+1)^2}{144} - \dfrac{(y-4)^2}{25} = 1$

35. $(y+6)^2 - (x-2)^2 = 1$

36. $\dfrac{(y-1)^2}{1/4} - \dfrac{(x+3)^2}{1/9} = 1$

37. $9x^2 - y^2 - 36x - 6y + 18 = 0$

38. $x^2 - 9y^2 + 36y - 72 = 0$

39. $9y^2 - x^2 + 2x + 54y + 62 = 0$

40. $16y^2 - x^2 + 2x + 64y + 63 = 0$

41. $x^2 - 9y^2 + 2x - 54y - 107 = 0$

42. $9x^2 - y^2 + 54x + 10y + 55 = 0$

Finding the Standard Equation of an Ellipse   In Exercises 43–52, find the standard form of the equation of the ellipse with the given characteristics. *See Example 5.*

43. Vertices: $(0, 2), (4, 2)$; minor axis of length 2

44. Vertices: $(3, 1), (3, 9)$; minor axis of length 6

45. Foci: $(0, 0), (4, 0)$; major axis of length 8

46. Foci: $(0, 0), (0, 8)$; major axis of length 16

47. Center: $(2, -1)$; vertex: $\left(2, \frac{1}{2}\right)$;
minor axis of length 2

48. Center: $(4, 0)$; vertex: $(4, -4)$; minor axis of length 6

49. Center: $(3, 2)$; $a = 3c$; foci: $(1, 2), (5, 2)$

50. Center: $(0, 4)$; $a = 2c$; vertices $(-4, 4), (4, 4)$

51. Vertices: $(5, 0), (5, 12)$; endpoints of minor axis: $(0, 6), (10, 6)$

52. Vertices: $(-12, 0), (0, 0)$; endpoints of minor axis: $(-6, 3), (-6, -3)$

Finding the Standard Equation of a Hyperbola   In Exercises 53–60, find the standard form of the equation of the hyperbola with the given characteristics.

53. Vertices: $(2, 0), (6, 0)$; foci: $(0, 0), (8, 0)$

54. Vertices: $(2, 3), (2, -3)$; foci: $(2, 5), (2, -5)$

55. Vertices: $(4, 1), (4, 9)$; foci: $(4, 0), (4, 10)$

56. Vertices: $(-2, 1), (2, 1)$; foci: $(-3, 1), (3, 1)$

57. Vertices: $(2, 3), (2, -3)$;
passes through $(0, 5)$

58. Vertices: $(-2, 1), (2, 1)$;
passes through $(4, 3)$

59. Vertices: $(0, 2), (6, 2)$;
asymptotes: $y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$

60. Vertices: $(3, 0), (3, 4)$;
asymptotes: $y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$

Identifying Conics   In Exercises 61–68, identify the conic by writing the equation in standard form. Then sketch its graph.

61. $x^2 + y^2 - 6x + 4y + 9 = 0$

62. $x^2 + 4y^2 - 6x + 16y + 21 = 0$

63. $4x^2 - y^2 - 4x - 3 = 0$

**64.** $y^2 - 4y - 4x = 0$

65. $4x^2 + 3y^2 + 8x - 24y + 51 = 0$

66. $4y^2 - 2x^2 - 4y + 8x - 15 = 0$

67. $25x^2 - 10x - 200y - 119 = 0$

68. $4x^2 + 4y^2 - 16y + 15 = 0$

69. Satellite Orbit   A satellite in a 100-mile-high circular orbit around Earth has a velocity of approximately 17,500 miles per hour. When this velocity is multiplied by $\sqrt{2}$, the satellite has the minimum velocity necessary to escape Earth's gravity, and it follows a parabolic path with the center of Earth as the focus (see figure).

Not drawn to scale
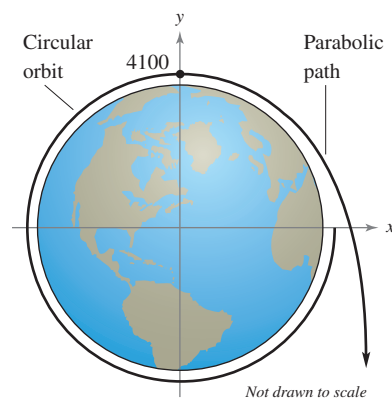
(a) Find the escape velocity of the satellite.

(b) Find an equation of its path (assume the radius of Earth is 4000 miles).

70. Fluid Flow   Water is flowing from a horizontal pipe 48 feet above the ground. The falling stream of water has the shape of a parabola whose vertex (0, 48) is at the end of the pipe (see figure). The stream of water strikes the ground at the point $\left(10\sqrt{3}, 0\right)$. Find the equation of the path taken by the water.
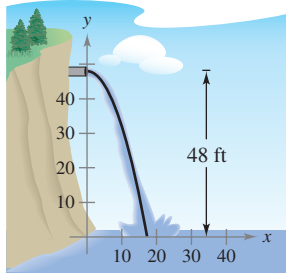
Eccentricity of an Ellipse   In Exercises 71–78, the flatness of an ellipse is measured by its **eccentricity** e, defined by

$$e = \frac{c}{a}, \text{ where } 0 < e < 1.$$

When an ellipse is nearly circular, e is close to 0. When an ellipse is elongated, e is close to 1 (see figures).
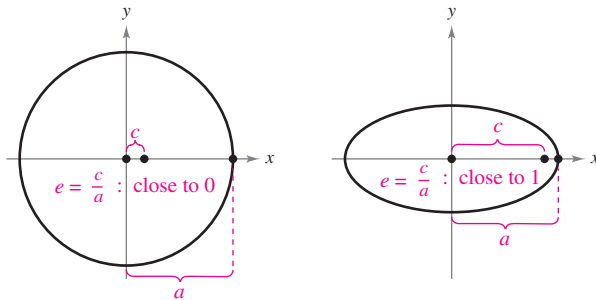
71. Find an equation of the ellipse with vertices $(\pm 5, 0)$ and eccentricity $e = \frac{3}{5}$.

72. Find an equation of the ellipse with vertices $(0, \pm 8)$ and eccentricity $e = \frac{1}{2}$.

73. Planetary Motion   Earth moves in an elliptical orbit with the sun at one of the foci (see figure). The length of half of the major axis is $a = 9.2956 \times 10^7$ miles and the eccentricity is 0.017. Find the shortest distance (perihelion) and the greatest distance (aphelion) between Earth and the sun.
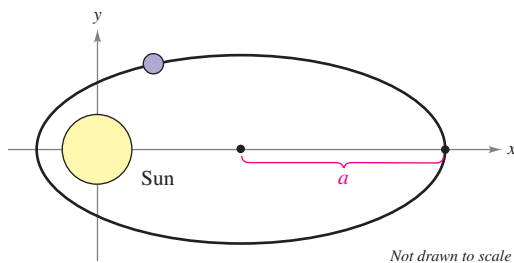

Figure for 73–75

74. Planetary Motion   The dwarf planet Pluto moves in an elliptical orbit with the sun at one of the foci (see figure). The length of half of the major axis is 3.670×10^9 miles and the eccentricity is 0.249. Find the shortest distance and the greatest distance between Pluto and the sun.

75. Planetary Motion   Saturn moves in an elliptical orbit with the sun at one of the foci (see figure). The shortest distance and the greatest distance between Saturn and the sun are 1.3495×10^9 kilometers and 1.5040×10^9 kilometers, respectively. Find the eccentricity of the orbit.

76. Satellite Orbit   The first artificial satellite to orbit Earth was Sputnik I (launched by the former Soviet Union in 1957). Its highest point above Earth's surface was 588 miles, and its lowest point was 142 miles (see figure). Assume that the center of Earth is one of the foci of the elliptical orbit and that the radius of Earth is 4000 miles. Find the eccentricity of the orbit.
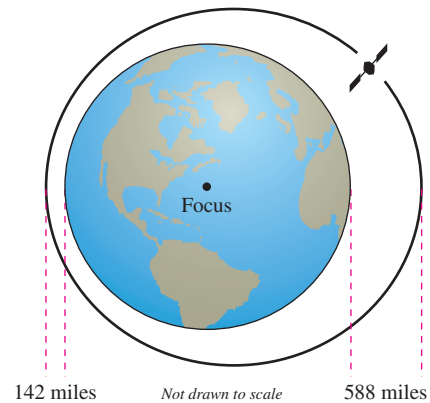
77. Alternate Form of Equation of an Ellipse   Show that the equation of an ellipse can be written as

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{a^2(1 - e^2)} = 1.$$

78. Comet Orbit   Halley's comet has an elliptical orbit with the sun at one focus. The eccentricity of the orbit is approximately 0.97. The length of the major axis of the orbit is approximately 35.88 astronomical units. (An astronomical unit is about 93 million miles.) Find the standard form of the equation of the orbit. Place the center of the orbit at the origin and place the major axis on the x-axis.

79. Australian Football   In Australia, football by *Australian Rules* (or rugby) is played on elliptical fields. The fields can be a maximum of 170 yards wide and a maximum of 200 yards long. Let the center of a field of maximum size be represented by the point (0, 85). Find the standard form of the equation of the ellipse that represents this field.   *(Source: Australian Football League)*